

Modeling Randomness

Common Core State Standards

S-MD.B.6 Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). **Also S-MD.B.7**

MP 1, MP 3, MP 4

Objective To understand random numbers
To use probabilities in decision-making



Think of as many ways as you can.



Getting Ready!

A class of 25 students wants to choose 4 students at random to bring food for a class party to be used in the platter shown. Any set of 4 students should have an equal chance of being chosen. What are some ways that the class could determine a random selection of 4 students?

In the Solve It, you thought of ways to select 4 people. In this lesson, you will learn ways to make a random selection and use probabilities to make decisions and solve real world problems.

Lesson Vocabulary

- expected value

Essential Understanding You can use probability to make choices and to help make decisions based on prior experience.

A random event has no predetermined pattern or bias toward one outcome or another. You can use random number tables or randomly generated numbers using graphing calculators or computer software to help you make fair decisions. In Reference 5 of this book, you will find a random number table to use with this lesson.

Think

How can a single-digit number be represented by two digits?

A zero and a number between one and nine can represent a single-digit number. For example, 09 can represent 9.



Problem 1 Making Random Selections

There are 28 students in a homeroom. Four students are chosen at random to represent the homeroom on a student committee. How can a random number table be used to fairly choose the students?

Step 1 Select a line from a random number table.

18823 18160 93593 67294 09632 62617 86779

Step 2 Group the line from the table into two digit numbers.

18 82 31 81 60 93 59 36 72 94 09 63 26 26 17 86 77 9

Step 3 Match the first four numbers less than 28 with the position of the students' names on a list. Duplicates and numbers greater than 28 are discarded because they don't correspond to any student on the list.

18 82 31 81 60 93 59 36 72 94 09 63 26 26 17 86 77 9

The students listed 18th, 9th, 26th, and 17th on the list are chosen fairly.



Got It? 1. A teacher wishes to choose three students from a class of 25 students to raise the school's flag. What numbers should the teacher choose based on this line from a random number table?

65358 70469 87149 89509 72176 18103 55169 79954 72002 20582



Problem 2 Making a Simulation

A cereal company is having a promotion in which 1 of 6 different prizes is given away with each box. The prizes are equally and randomly distributed in the boxes of cereal. On average, how many boxes of cereal will a customer need to buy in order to get all 6 prizes?

Plan

How do you start to model a situation?

The first step in modeling a situation is determining the possible outcomes and assigning a number to each outcome.

Step 1 Let the digits from 1 to 6 represent the six prizes.

Step 2 Use a graphing calculator and enter the function $\text{randInt}(1,6)$ to generate integers from 1 to 6 to simulate getting each prize. One trial is completed when all 6 digits have appeared. The circled numbers represent the first appearance of each number.

2 5 3 2 5 2 2 4 5 1 2 2 4 4 5 6

Step 3 Count how many boxes of cereal will be bought before all the digits 1 through 6 have appeared.

In this trial, all six prizes were collected after the purchase of the 16th box of cereal.

Step 4 Conduct additional trials. For 19 more simulations the results were: 17, 12, 21, 17, 8, 11, 14, 10, 16, 23, 19, 15, 9, 27, 20, 10, 18, 12, 13. For the 20 simulations, it will take, on average, $\frac{308}{20} = 15.4$ boxes of cereal to get all 6 prizes.



Got It? 2. Suppose that to win a game, you must roll a standard number cube until all of the sides show at least one time. On average, how many times will you have to roll the number cube before each side shows at least one time? Use a random number table or a random number generator.

In Problem 2, you figured out how many boxes of cereal on average you would expect to have to buy to collect all 6 prizes. You can use this information to make a decision about whether it is worth trying to collect all 6 prizes. Similarly, you can use the *expected value* of a situation that involves uncertainty to make a decision. *Expected value* uses theoretical probability to tell you what you can expect in the long run. If you know what *should* happen mathematically, you make better decisions in problem situations. The **expected value** is the sum of each outcome's value multiplied by its probability.

take note

Key Concept Calculating Expected Value

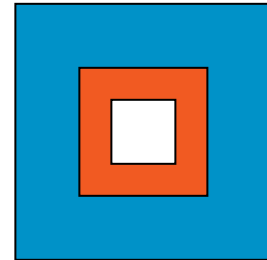
If A is an action that includes outcomes A_1, A_2, A_3, \dots and $\text{Value}(A_n)$ is a quantitative value associated with each outcome, the expected value of A is given by

$$\text{Value}(A) = P(A_1) \cdot \text{Value}(A_1) + P(A_2) \cdot \text{Value}(A_2) + \dots$$



Problem 3 Calculating an Expected Value

Suppose you are at a carnival and are throwing darts at a board like the one at the right. There is an equally likely chance that your dart lands anywhere on the board. You receive 20 points if your dart lands in the white area, 10 points if it lands in the red area, and -5 points if it lands in the blue area. How many points can you expect to get given that the areas for each region are white, 36 in.²; red, 108 in.²; and blue, 432 in.²? The total area is 576 in.².



Value(throw)

$$\begin{aligned} &= P(\text{white area}) \cdot (\text{white points}) + P(\text{red area}) \cdot (\text{red points}) + P(\text{blue area}) \cdot (\text{blue points}) \\ &= \frac{36}{576} \cdot 20 + \frac{108}{576} \cdot 10 + \frac{432}{576} \cdot (-5) \quad \text{Substitute the ratio of each area to the total area} \\ &\quad \text{and the points for each section.} \\ &= 1.25 + 1.875 + (-3.75) \quad \text{Multiply.} \\ &= -0.625 \quad \text{Simplify.} \end{aligned}$$

You can expect to get -0.625 points.



- Got It?** 3. a. How many points can you expect to get for the dart board shown in Example 3 if you receive these points: 30 points for white, 15 points for red, and -10 points for blue?
- b. For part (a), what number of points could be set for the blue area that would result in an expected value of 0?

Expected values can be used to make data-driven decisions. You can calculate the expected values of situations of interest, and compare them to decide which is most favorable.

Think

How can a coach know the probability for his team's kicking a field goal?

He can know the experimental probability based on how many successful field goals divided by the number of attempts in similar situations.



Problem 4 Making Decisions Based on Expected Values

Football On the opening drive, a football coach must decide whether to kick a field goal (FG) or go for a touchdown (TD). The probabilities for each choice based on his team's experience are shown on the page from his playbook at the right. A field goal will give his team 3 points. His team will get 7 points if the touchdown is successful. Which play should he choose?

	Pts.	Prob.
FG	3	90%
TD	7	35%

Step 1 Calculate the expected value of both plays.

Field Goal: $90\% \cdot 3 = 2.7$ points

Touchdown: $35\% \cdot 7 = 2.45$ points

Step 2 Choose the play with the greater expected value.

$2.7 > 2.45$

The coach should choose the field goal.



- Got It?** 4. a. Suppose the probability for the field goal was 80% and the probability for a touchdown was 30%. Which play should the coach choose?
b. Are there situations where the coach should choose a play that doesn't have the greatest expected value? Explain.



Lesson Check

Do you know HOW?

- What are the first four numbers between 1 and 45 which would be chosen on the basis of the following line from a random number table?
81638 36566 42709 33717 59943 12027 46547
- A basketball player can either attempt a 3-point shot (with a 25% probability of scoring) or pass to a teammate with a 50% probability of scoring 2 points. What are the expected values for each choice? Which choice should he make?

Do you UNDERSTAND?



MATHEMATICAL PRACTICES

- Reasoning** A friend says that using a random number table to pick a person at random isn't as fair as throwing a dart at the list of names. Explain your friend's error.
- Vocabulary** Explain the meaning of expected value. Include an example.
- Writing** Describe how you can use a random number generator to model the results of tossing a coin a hundred times.



Practice

Use the lines from a random number table to select numbers to use in each problem.

← See Problem 1.

6. Choose 3 students from a list of 45 volunteers.
72749 13347 65030 26128 49067 27904 49953 74674 94617 13317
7. Choose 5 families to survey from a phone directory page with 950 names.
11873 57196 32209 67663 07990 12288 59245 83638 23642 61715

For Exercises 8 and 9, describe how to use random numbers to do a simulation for each situation.

← See Problem 2.

8. A teacher assigns students new seats randomly each week. On average, how long will it be before a student is assigned the same seat for two weeks in a row?
9. **Sports** A basketball player makes 80% of her free throw attempts. Find the average number of free throw attempts needed in order to make 3 free throws in a row.
10. **Games** In a game show, a contestant receives a prize that has a 5% probability of being worth \$1000 and a 95% probability of being worth \$1. What is the expected value of winning a prize?
11. **Video Games** In a video game, a player has a 80% probability of winning 1000 points if he attacks a certain monster. There is a 20% probability that he will lose 25,000 points. What is the expected value of points earned? Should the player attack the monster? Explain.

← See Problems 3 and 4.



Apply



12. **Think About a Plan** A business owner is choosing which of two potential products to develop. Developing Product A will cost \$10,000, and the product has a 60% probability of being successful. Product B will cost \$15,000 to develop, and it has a 30% probability of being successful. If Product A is successful, the business will gain \$200,000. If product B is successful, the business will gain \$450,000. Which product should the business owner choose?
 - How can the expected values of Product A and Product B help the business owner make the decision?
 - Compare the expected values. Recommend the business owner choose the product with the greater expected value.
13. **Business** A company is deciding whether to invest in a new business opportunity. There is a 40% chance that the company will lose \$25,000, a 25% chance that the company will break even, and a 35% chance that the company will make \$40,000. Should company executives decide to invest in the opportunity? Explain.
14. **Finance** A stock has a 25% probability of increasing by \$10 per share over the next month. The stock has a 75% probability of decreasing by \$5 per share over the same period. What is the expected value of the stock's increase or decrease? Based on this information, should you buy the stock? Explain.

15. **Test Taking** You earn 1 point for each correct response on a multiple choice test and lose 0.5 point for an incorrect response. Each question has four answer choices. If a student does not know the correct answer and guesses, what is the expected value of the guess?
16. **Games** A bag contains 10 marbles; 4 are red, 5 are yellow, and 1 is blue. You draw one marble from the bag without looking. If you draw a blue marble, you win \$10. If you draw a red marble, you win \$5, and if you draw a yellow marble, you win \$3. What is the expected value of drawing one marble? Would you play this game for \$5? Explain.
- © 17. **Writing** A polling company has been hired to survey 500 households out of a possible 2500 households. How can the 500 households be randomly selected for the survey?



Challenge

18. **Stocks** An investor is choosing between three stocks she might purchase. The potential outcomes are listed in the table.
- Calculate the expected value of the gain or loss for each stock.
 - Set up a model and use a random number table to model the investment in each stock.
 - How do the results of the random number table model compare with the expected values?
 - Does the expected value tell you anything about the riskiness of each stock?

Potential Outcomes

	Lose \$25 per share	Gain \$5 per share	Gain \$45 per share
Stock ABC	40%	15%	45%
Stock JKL	15%	65%	20%
Stock MNO	5%	80%	15%

Standardized Test Prep



SAT/ACT

19. What is the sum of the interior angles of an octagon?
- (A) 135 (B) 360 (C) 480 (D) 1080
20. A kite has an 80° angle and a 50° angle. Which of the following might be the measure of the remaining angles?
- (A) 80 and 50 (B) 50 and 180 (C) 80 and 150 (D) 50 and 150



Short Response

21. What are the lengths of the two legs of a right triangle with a 60° angle and a hypotenuse 1 meter long? Round your answer to the nearest centimeter.

Mixed Review

Calculate the following conditional probabilities.

↩ See Lesson 13-5.

22. The conditional probability $P(A | B)$ when $P(B) = 30\%$ and $P(A \text{ and } B) = 20\%$
23. The conditional probability $P(A | B)$ when $P(B) = 75\%$ and $P(A \text{ and } B) = 50\%$

What is the standard equation of each circle?

↩ See Lesson 12-5.

24. center $(3, -5)$; radius, 9

25. center $(-2, 8)$; radius, 5